FUNCTIONAL CALCULUS AND A LINK TO FRACTIONAL CALCULUS

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Dedicated to Francesco Mainardi on the occasion of his 60 th birthday

Abstract

In this paper we sketch the state of the art of our *functional calculus* approach to non-integer differentiation. In particular, its generalisation to distributional spaces is described. A generalisation of the law for differentiating convolutions is used to show that this approach coincides for certain classes of "admissible" functions with the definition via *Riemann–Liouville* integrals as well as via *Caputo* integrals with lower bounds $-\infty$.

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1. Introduction

1.1. Survey

In 1995 two of the authors (*Beyer, Kempfle*) introduced in [1] a functional calculus definition of non-integer pseudo-differential operators along a model of a viscously damped one–mass–oscillator. It is based on the unitary *Fourier* transformation on the space of square–summable functions on the real line. Up to now and documented in several papers (e.g. [1, 21, 22]) the approach has been extended to distributional spaces (briefly sketched below). The behaviour of the solutions of related differential equations was investigated, particularly in view of causality and stability. This was possible by using fully developed instruments of functional analysis.

The functional calculus approach has the advantage of a well understood and mathematically stringent background which in the meanwhile has become popular through the theory of pseudo-differential operators ([14, 6, 19]). Results like the *"Closed graph theorem"* or the theorems of *Hahn–Banach* up to those of *Paley–Wiener* can be used to describe the a priori scope and behaviour of solutions of related differential equations.

A comparison with the most used fractional calculi was given in [15]. In case of finite lower bounds the a priori definitions via integral–operators (*Riemann–Liouville, Caputo*) have disadvantages like a missing semigroup property as well as missing translation invariance, properties which are present ex definition in the functional calculus approach (see below Remark 3.1.1).

The question of the posing initial conditions in hereditary systems was considered in [20]. We have shown that in fractional models the past cannot be represented by a finite number of initial conditions, i.e., the whole past is needed as a "global initial condition".

Meanwhile we have verified our approach experimentally ([16, 25, 24, 21]). Measurements were done on the frequency responses of visco-elastic rods made of Teflon, Poly-Ethylen, Poly-Urethan, Poly-Vinyl-Chlorid etc. as well as on impulse responses of one–mass–oscillators of the same materials. All measurements show very good agreement to our mathematical model within a wide frequency domain in case of rods ($\leq$ 12000 Hz) and a wide range of masses in case of the one–mass–oscillator ($\leq$ 10 kg). We emphasise once more that the steady state modelling of frequency responses requires an approach which includes all the past.

But, coming back to the question of comparing of different approaches, there is some common ground:

In case of linear fractional systems which are at rest up to some time $t = 0$ the usual *Riemann–Liouville* approach, the modified *Caputo* approach and our *functional calculus* approach give the same impulse responses.

We will show at the end of this paper more precisely the relationsship between those three approaches.

Before we remind the reader on the well–known definitions of the most common fractional calculi we introduce the following notation (see also [10–13]).

1.2. Notations

1. The symbol i denotes the imaginary unit, the real and imaginary part of some $s \in \mathbb{C}$ are denoted by $\Re(s)$ and $\Im(s)$, respectively. D is the general derivative operator.

2. As commonly done, we write $\mathbf{L}_2 := \mathbf{L}_2(\mathbb{R})$ for the set of square–summable complex–valued functions as well as for the vector space of its equivalence classes. Consequently, we write $g(t) = f(t)$ without "a.e.".

3. We write $\mathcal{S}$ for the *Schwartz* space of rapidly decreasing $\mathbf{C}^\infty$–functions $(\mathbb{R} \to \mathbb{C})$, $\mathcal{D}$ denotes the test space of $\mathbf{C}^\infty$–functions with compact support. The dual spaces $\mathcal{D}'$, $\mathcal{S}'$ of continuous linear functionals $f : \varphi \mapsto \langle f , \varphi \rangle$, $\varphi \in \mathcal{D}, \mathcal{S}$ are called "space of *distributions*" ($\mathcal{D}'$) and "space of *tempered distributions*" ($\mathcal{S}'$). Additionally, the space of all $\mathcal{D}'$–elements with compact support is denoted by $\mathcal{E}'$.

The *Dirac* impact belongs to $\mathcal{S}'$, $\mathcal{D}'$ and $\mathcal{E}'$ and is defined via $\langle \delta , \varphi \rangle := \varphi(0)$. But as common in physics and engineering we do not stress the functional notation and use generalised functions without any brackets, if the meaning is clear, e.g. $\delta(t)$, $\theta^{(n)}(t)$, etc.

Concerning the important discussion of the usual topologies in which all considered spaces are complete topological vector spaces with the dense embeddings

$$\mathcal{D} \Subset \mathcal{S} \Subset \mathbf{L}_2 \Subset \mathcal{S}' \Subset \mathcal{D}'$$

we refer e.g. to [26].

4. The *Fourier* transformation is denoted by $\mathcal{F}$, the *Fourier* transform of some function $x(t)$ is denoted by $\widehat{x}(\omega)$.

5. Finally, all noninteger powers s^q $(s \in \mathbb{C})$ are defined as principal branches, more precisely and in accordance with nearly all computer algebraic systems (Mathematica, Maple, Mathlab, etc.), we set $-\pi < \arg(s) \leq \pi$.

2. Riemann–Liouville and Caputo derivatives

2.1. Riemann–Liouville derivatives

Definition 2.1. *Let* $a \geq -\infty,\ q > 0,\ \ q \notin \mathbb{N}$.
Firstly, we define a fractional integral of order q *of a function* $x(t),\ t \geq a$ *by*

$$ {}_aI_t^q\, x(t) := \frac{1}{\Gamma(q)} \int_a^t \frac{x(\tau)}{(t-\tau)^{1-q}}\, d\tau\,. \tag{1}$$

Then let $n \in \mathbb{N},\ n-1 < q < n$ *and define the RL–derivative of order* q *as*

$$\begin{aligned} {}_aD_t^n x(t) &:= D^n x(t) := x^{(n)}(t)\,, \\ {}_aD_t^q\, x(t) &:= D^n\, {}_aI_t^{n-q}\, x(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int\limits_a^t \frac{x(\tau)}{(t-\tau)^{q-n+1}}\, d\tau\,. \end{aligned} \tag{2}$$

For theoretical treatment it is only relevant, whether $a = -\infty$ or not. Thus we distinguish only the two cases

$$ {}_{-\infty}D_t^q \quad \text{and} \quad {}_0D_t^q\ ,\ \text{respectively.}$$

2.2. The Caputo definition

Caputo ([3]) has suggested a modification of Definition 1.1 exchanging integration and differentiation in (2)

Definition 2.2.

$$ {}_a^C D_t^q\, x(t) := {}_aI_t^{n-q} D^n\ x(t) = \frac{1}{\Gamma(n-q)} \int\limits_a^t \frac{x^{(n)}(\tau)}{(t-\tau)^{q-n+1}}\, d\tau \tag{3}$$

if $n-1 < q < n$ *and* ${}_a^C D_t^q\, x(t) := x^{(n)}(t)$, *if* $q = n$.

Gorenflo and *Mainardi* ([4]) have regularised the discontinuities of this derivative for $a = 0$ at integer n by modifications of the considered differential equations. The relation between the two definitions is namely (see [4], (1.19) or [9], (2.16)):

$$ {}_aD_t^q\, x(t) = {}_a^C D_t^q\, x(t) + \sum_{k=0}^{n-1} \frac{t^{k-q}}{\Gamma(k-q+1)}\, x^{(k)}(a_+) \tag{4}$$

3. Our Functional Calculus approach in $\mathbf{L_2}$

3.1. The definition

DEFINITION 3.1. *Let $\mathcal{F}$ denote the unique $\mathbf{L}_2$–extension of*

$$\widetilde{\mathcal{F}}: \mathbf{L}_1 \cap \mathbf{L}_2 \to \mathbb{C}: \; x(t) \mapsto \widehat{x}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-i\omega t} x(t)\, \mathrm{d}t$$

Then a fractional differential operator D^q, $(q \geq 0)$ is defined in $\mathbf{L}_2$ via

$$D^q := \mathcal{F}^{-1} \, (i\omega)^q \, \mathcal{F} \, ,$$

such that the q–th derivative of $x(t) \in \mathbf{L}_2$ is given as

$$D^q x(t) = \mathcal{F}^{-1} \left\{ (i\omega)^q \, \widehat{x}(\omega) \right\} \, ,$$

if $(i\omega)^q \, \widehat{x}(\omega) \in \mathbf{L}_2$.

REMARKS 3.1. **1.** The so–called *symbol* $(i\omega)^q$ is defined (see **1.2.5.**) as

$$(i\omega)^q = |\omega|^q \, \mathrm{e}^{i \,\mathrm{sign}(\omega)\, q\, \pi/2}$$

From this one easily verifies the semigroup property of the class of this symbols and thus of the fractional derivatives via

$$(i\omega)^q \, (i\omega)^p = |\omega|^q \, \mathrm{e}^{i \,\mathrm{sign}(\omega)\, q\, \pi/2} \cdot |\omega|^p \, \mathrm{e}^{i \,\mathrm{sign}(\omega)\, p\, \pi/2} = |\omega|^{q+p} \, \mathrm{e}^{i \,\mathrm{sign}(\omega)\, (q+p)\, \pi/2} = (i\omega)^{q+p}$$

2. The integer derivatives are included trivially: the above definition is a generalisation of the differentiation rule of *Fourier* transforms

3. As treated in previous papers linear combinations of those symbols/deriva tives are defined in their entirety as one symbol/operator. The solutions of related differential equations can then be analysed from the roots of the symbol ([2, 21]).

4. It is well–known that the range of all $x(t) \in \mathbf{L}_2$, such that $(i\omega)^q \, \widehat{x}(\omega) \in \mathbf{L}_2$ is a dense subset of $\mathbf{L}_2$. This way D^q becomes densely defined, linear and closed. The last property is via the *Closed graph theorem* responsible for the continuity of related solutions.

3.2. The limits of the $\mathbf{L}_2$–approach

For many applications, particularly in solid dynamics, the $\mathbf{L}_2$–approach comes out satisfactory ([1, 2]). We sketch briefly previous results:

Let $\mathcal{A} = \sum_{k=0}^{N} a_k D^{q_k}$, $q_k \in \mathbb{R}^+$, $a_k \in \mathbb{R}$ be a linear (pseudo) differential operator and (via $i\omega \mapsto s$) $A(s) = \sum_{k=0}^{N} a_k s^{q_k}$ its symbol. Then the $\mathbf{L}_2$–solution

$$\mathcal{A}\, x(t) = f(t) \tag{5}$$

can be represented as a convolution with the so called impulse response $K(t)$, i.e.,

$$x(t) = (K * f)\,(t) \, , \text{ where } \; K(t) = \frac{1}{\sqrt{2\pi}} \int_{-i\infty}^{i\infty} \frac{\mathrm{e}^{st}}{A\,(s)} \, \mathrm{d}s. \tag{6}$$

It holds:

1. If the largest order of derivatives ($q_N = \deg \mathcal{A}$) is > 1, then $K(t)$ is always an continuous $\mathbf{L}_2$–function.
2. For $\frac{1}{2} < \deg \mathcal{A} \leq 1$, $K(t)$ is continuous on $\mathbf{L}_2(\mathbb{R}\backslash\{0\})$ with finite jump at $t = 0$.
3. For $0 < \deg \mathcal{A} \leq \frac{1}{2}$ the jump degenerates into a pole.
4. If there are no zeros s_k of $A(s)$ with $\Re(s_k) \geq 0$ (the stable case), then we get from a *Paley–Wiener* theorem that the impulse response and thus the solutions of (5) are causal, i.e., $K(t) = 0$ for $t < 0$ and hence $f(t) = 0$ for $t < t_0$ causes $x(t) = 0$ for $t < t_0$.
5. Thus stable linear systems in solid mechanics can in principle be treated without any problems. Considering the so–called unstable case, i.e., there are zeros in the right halfplane, (6) delivers consequently via items 1.,2.,3. a noncausal $K(t)$, such that the convolution would yield a noncausal but stable "solution".

We depict the situations from items 4., 5. in figure 1.

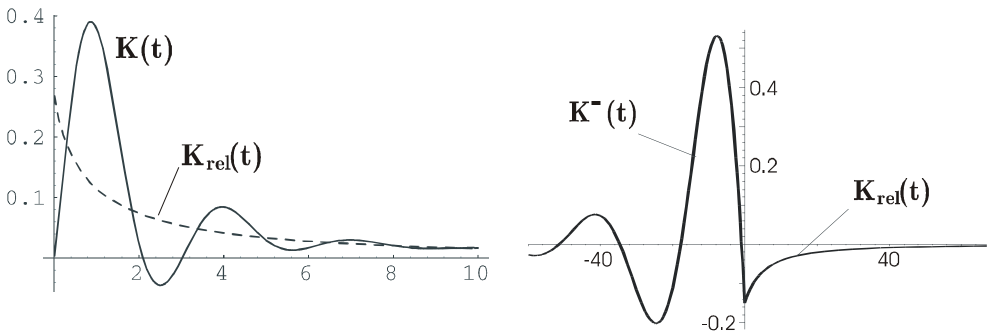

Figure 1. Causal and noncausal impulse responses.

The left graph belongs to the operator $D^2 + 3\,D^{0.6} + 1$, the right hand one to $5\,D^2 - D^{0.1} + 1$. Here $\mathbf{K}^-$ denotes the via residue theorem resulting part for $t < 0$. In both graphs $\mathbf{K}_{\mathrm{rel}}$ denotes the slow relaxations which dominate asymptotically the ($\mathbf{L}_2$–) impulse responses for $t \to \infty$. The order of decay is $O(t^{-1-q})$, if q is the smallest noninteger order of derivatives in the operator (In our examples 0.6 and 0.1, resp.)

Observation 3.2. *Obviously, the right hand picture shows a mathematically correct L_2–solution. But, a physically consistent impulse response has to be causal. Thus we conclude from linear operator theory that $\mathbf{K}^-(\mathbf{t})$ is part of a kernel function of $\mathcal{A}$, which obviously is not a L_2–function, because the continuation of $\mathbf{K}^-$ on $\mathbb{R}$ turns out to be an exponentially increasing oscillation. Thus the physically correct solution is the superposition of the depicted noncausal impulse response with just this "homogeneous" solution of $\mathcal{A}x = 0$. Truly, to justify mathematically this heuristic proceeding, we have to extend the scope of our approach, such that exponentially increasing functions (and thus the kernels of integer order operators) are included. A convenient function space in view to Fourier analysis is $\mathcal{D}'$.*

Before we sketch this extension we finish our illuminative example with the resulting causal but unstable impulse response of the just discussed case which is depicted in figure 2.

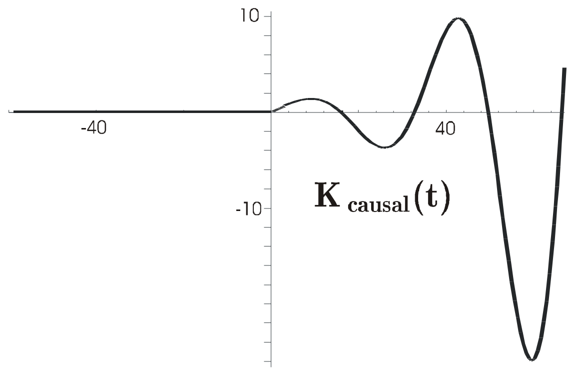

Figure 2. Causal unstable impulse response.

4. Extension on distributional spaces

For brevity we put the main focus on the basic ideas and omit technical details (We have described the following embeddings in [21] and [22]). The procedure consists of two steps. The first extends the Fourier transformation, in the second step we extend our functional calculus by establishing the symbol as an admissible multiplicator.

4.1. Fourier transforms in distributional spaces

4.1.1. Tempered distributions

The test space $\mathcal{S}$ consists briefly of all those $\mathbf{C}^{\infty}$–functions which decay faster than any integer negative power t^{-n} for $|t| \to \infty$. Consequently, the regular elements of the dual space $\mathcal{S}'$ of all continuous linear functionals on $\mathcal{S}$ can just be identified with those locally integrable functions $\mathbb{C} \to \mathbb{R}$ which have at most polynomial increasement. Hence it is not the final space we are looking for, though there is an easy established Fourier analysis. As exposed in [18, §3] one gets via a complete metric on $\mathcal{S}$

Result 4.1.

1. *The Fourier transformation acts as an unitary isomorphism on* S *:*

$$\mathcal{F}(\mathcal{S}) = \mathcal{F}^{-1}(\mathcal{S}) = \mathcal{S}$$

2. *If the Fourier transformation on* S' *is defined as*

$$\widehat{f} := \langle f \, , \, \widehat{\varphi} \rangle \quad \text{for all} \;\; \varphi \in \mathcal{S} \; ,$$

then item 1. holds also on S' :

$$\mathcal{F}(\mathcal{S}') = \mathcal{F}^{-1}(\mathcal{S}') = \mathcal{S}'$$

4.1.2. Distributions

Unfortunately there is no such simple result on $\mathcal{D}$ and $\mathcal{D}'$. The well–known correspondence $\widehat{\delta}(\omega) = \sqrt{2\pi}$ shows drastically that the bounded support is not preserved by $\mathcal{F}$. On the other hand, $\mathcal{D}'$ has the properties we are looking for, because its regular elements can be identified with just the locally integrable functions $\mathbb{R} \to \mathbb{C}$ ([26], 6.1).

The characterisation of $\mathcal{F}(\mathcal{D})$ needs a loop way via an extension of the Fourier transforms $\widehat{x}(\omega)$ from $\omega \in \mathbb{R}$ to $\zeta \in \mathbb{C}$. More precisely

Definition 4.2. *The Fourier–Laplace transform of some $x(t) \in L_1 \cap L_2$ is defined as*

$$\mathcal{F}_{\mathbb{C}}\{x(t)\}(\zeta) := \widetilde{x}(\zeta) := \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \mathrm{e}^{-i\zeta t} x(t)\,\mathrm{d}t$$

$\widetilde{x}(\zeta)$ exists and is analytic at least in some open neighbourhood of the real axis (from the lemma of *Riemann–Lebesgues*). For the subspace $\mathcal{D} \subset L_1 \cap L_2$ there is even a *Paley–Wiener* theorem which is optimal for our purposes ([26], 7.2.2).

Theorem 4.2. *Let* $\mathbf{B}_r := \{\, |t| \le r \mid r > 0\}$.

For every $\phi \in \mathcal{D}$ with support in $\mathbf{B}_r$ it holds

1. $\widetilde{\phi}(\zeta)$, $\zeta = \xi + i\eta \quad (\xi, \eta \in \mathbb{R})$ *is entire.*
2. *There are constants $0 < c_k < \infty$ such that*

$$\left|\widetilde{\phi}(\zeta)\right| \le c_k \left(1 + |\zeta|\right)^{-k} e^{r|\eta|} \quad \forall\, k \in \mathbb{N}.$$

3. *Conversely, if an entire function $\psi(\zeta)$ satisfies 2., then there exists $\phi \in \mathcal{D}$, with support in $\mathbf{B}_r$, such that 1. holds, i.e*

$$\psi(\zeta) = \widetilde{\phi}(\zeta), \quad \zeta \in \mathbb{C}.$$

Definition 4.1. *We denote by $\mathcal{Z}$ the space of all entire functions for which there exist constants $r > 0$ such that item 2. of theorem 4.2 holds.*

The space of all restrictions on $\mathbb{R}$ is briefly denoted by $\mathcal{Z}|_{\mathbb{R}}$.

Remarks 4.1.

1. From Definition 4.2 we conclude firstly $\widehat{x}(\xi) = \widetilde{x}(\zeta)|_{\mathbb{R}}$, $(\zeta = \xi + i\eta)$. Secondly and vice versa, the *Paley–Wiener* theorem reveals via analytic continuation that $\widetilde{x}(\zeta)$ is uniquely determined by $\widehat{x}(\xi)$. Together with item 3 we recognise that $\mathcal{F}_{\mathbb{C}}$ provides a 1–1–mapping from $\mathcal{D}$ on $\mathcal{Z}$ such that $\mathcal{F}$ provides a 1–1–mapping from $\mathcal{D}$ on $\mathcal{Z}|_{\mathbb{R}}$. Briefly we have

$$\mathcal{F}(\mathcal{D}) = \mathcal{F}^{-1}(\mathcal{D}) = \mathcal{Z}|_{\mathbb{R}} \quad \text{and} \quad \mathcal{F}(\mathcal{Z}|_{\mathbb{R}}) = \mathcal{F}^{-1}(\mathcal{Z}|_{\mathbb{R}}) = \mathcal{D}. \tag{7}$$

2. To understand the afterwards following construction of Fourier transforms on $\mathcal{D}'$ we have to point out that the topology of $\mathcal{Z}$ is induced via $\mathcal{F}_{\mathbb{C}}$ from the

topology of $\mathcal{D}$. Via item 1 this comes out the same as to extend the topology of $\mathcal{Z}|_{\mathbb{R}}$ which is induced from the $\mathcal{D}$–topology via $\mathcal{F}$. (Note that the supports of all test spaces are subspaces of $\mathbb{R}$, such that we cannot use the usual $\mathbb{C}$–topology). Hence $\mathcal{Z}$ as well as $\mathcal{Z}|_{\mathbb{R}}$ are ex construction complete topological vector spaces, moreover via item 1 from theorem 4.2 and $\mathcal{Z}|_{\mathbb{R}} \Subset \mathcal{S}$, both are admissible test spaces.

3. Defining now $\mathcal{Z}' = \{ f := \langle f \,,\, \psi \rangle \mid \psi \in \mathcal{Z} \}$ and $\mathcal{Z}|_{\mathbb{R}}' = \{ g := \langle g \,,\, \varphi \rangle \mid \varphi \in \mathcal{Z}|_{\mathbb{R}} \}$ we conclude from item 2 that $\mathcal{Z}'|_{\mathbb{R}} = \mathcal{Z}|_{\mathbb{R}}'$.

   We may further complete the chain of embeddings given in 0.3.

$$\mathcal{D} \Subset \mathcal{Z}|_{\mathbb{R}} \Subset \mathcal{S} \Subset \mathbf{L}_2 \Subset \mathcal{S}' \Subset \mathcal{Z}|_{\mathbb{R}}' \Subset \mathcal{D}'$$

With respect to (7) we give now finally the extension of $\mathcal{F}$:

Definition 4.2. *We define the Fourier transforms of distributions via*

1. $\mathcal{F}: \ \mathcal{D}' \to \mathcal{Z}|_{\mathbb{R}}' : f \mapsto \widehat{f} = \left\langle \widehat{f} \,,\, \phi \right\rangle := \left\langle f \,,\, \widehat{\phi} \right\rangle \quad \text{for all} \quad \phi \in \mathcal{D}$
2. $\mathcal{F}: \ \mathcal{Z}|_{\mathbb{R}}' \to \mathcal{D}' : g \mapsto \widehat{g} = \langle \widehat{g} \,,\, \varphi \rangle := \langle g \,,\, \widehat{\varphi} \rangle \quad \text{for all} \quad \varphi \in \mathcal{Z}|_{\mathbb{R}}'$

4.2. Fractional derivatives on distributional spaces

The "problem" in the extension of the $\mathbf{L}_2$–definition of a fractional derivative, i.e., $D^q x(t) = \mathcal{F}^{-1}\{(i\omega)^q\, \widehat{x}(\omega)\}$ is based in the construction of "operations in terms of distributions", namely, this terminus implies that operations are defined in the test space. But the product $(i\omega)^q\, \varphi(\omega),\ \varphi \in \mathbf{C}^\infty,\ q \notin \mathbb{N}$ is not in $\mathbf{C}^\infty$. Truly, one recognises at first glance that this can only be a technical problem, because this product is well–defined on all compact subspaces of $\mathbb{R}\backslash\{0\}$ and $(i\omega)^q$ vanishes continuously at the only critical point $\omega = 0$. Moreover, the product of all regular distributions in $\mathcal{S}'$, $\mathcal{D}'$, $\mathcal{E}'$ with $(i\omega)^q$ in terms of the "normal", i.e., pointwise product remains regular. Now, by completeness of the distributional spaces all elements can be approximated in terms of distributions by regular ones. Hence all definitions can only differ by some point measure in the origin. If we now demand with respect to physical consistency further that continuous functions should keep their continuity in the origin, we arrive at the following natural definition.

Definition 4.3. *Let $\mathcal{K}$ denote one of the spaces $\mathcal{S}$, $\mathcal{D}$, $\mathcal{E} := \mathbf{C}^\infty$ and let $f \in \mathcal{K}'$. Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{K}$ be a regular sequence that converges in $\mathcal{K}$ to f, i.e.:*

$$\langle f_n \,,\, \varphi \rangle \to \langle f \,,\, \varphi \rangle \quad \forall\, \varphi \in \mathcal{K}$$

Then $(\langle (i\omega)^q\, f_n \,,\, \varphi \rangle)_{n\in\mathbb{N}}$ is a regular sequence that converges in $\mathcal{K}'$, such that

$$(i\omega)^q\, f := \lim_{n\to\infty} \langle (i\omega)^q\, f_n \,,\, \varphi \rangle$$

is well–defined and establishes a physically consistent fractional derivative

$$D^q x(t) = \mathcal{F}^{-1}\{(i\omega)^q\, \widehat{x}(\omega)\} \quad \forall\, x \in \mathcal{K}\,.$$

Remarks 4.2.

1. The convergence of the sequence $\left((i\omega)^q f_n\right)_{n\in\mathbb{N}}$ follows by dominated convergence from the well–set definition of $(i\omega)^k g(\omega)$, $q < k \in \mathbb{N}$ for all $g \in \mathcal{K}'$.
2. We emphasise that contrary to the situation in $\mathbf{L}_2$, where $(i\omega)^q$ is an admissible multiplicator only on a dense subset, $(i\omega)^q$ is always an admissible $\mathcal{K}'$–multiplicator, such that all fractional derivatives exist in $\mathcal{K}'$.

To illustrate that this definition is a "natural extension" of integer derivatives we give some

Examples 4.1. ($\theta(t)$ denotes the *Heaviside* unit step)

1. $D^q\, e^{at+b} = a^q e^{at+b} \quad (a, b \in \mathbb{C})$
2. $D^q\left(t^m\, e^{at+b}\right) = e^{at+b} \sum_{k=0}^{m} \binom{m}{k} \dfrac{\Gamma(q+1)}{\Gamma(q+1-k)} a^{q-k} t^{m-k} \quad \begin{pmatrix} m \in \mathbb{N} \\ a \neq 0 \end{pmatrix}.$
3. $D^q\Big(e^{at} \sin(\sigma t)\Big) = e^{at} r^q \sin(\sigma t + q\vartheta) \quad (\sigma > 0),$

   where $r = \sqrt{a^2+\sigma^2}$; $\tan(\vartheta) = |a|/\sigma$.
4. $D^q\theta(t) = \dfrac{\theta(t)\, t^{-q-1}}{\Gamma(-q)} \quad \left(= D^{n-1}\delta(t), \text{ if } q = n \in \mathbb{N}\right).$

   Hence, in case of $q \notin \mathbb{N}$, we have no point measure!
5. $D^q\delta(t) = D^{q+1}\theta(t) = \dfrac{\theta(t)\, t^{-q}}{\Gamma(1-q)}$
6. $D^q\left\{t^p\theta(t)\right\} = \dfrac{\Gamma(1+p)}{\Gamma(1+p-q)}\theta(t)\, t^{p-q} \quad (0 < q < p)$
7. $D^q t^p = (-1)^q \dfrac{\sin(\pi q)\, \Gamma(1+p)}{\sin(\pi(p-q))\, \Gamma(1+p-q)}\, t^{p-q} \quad (0 < q < p;\ p, p-q \notin \mathbb{N})$

   This formula holds via $\sin(\pi p) = (-1)^m \sin(\pi(p-m))$ also for integer order differentiation, even if $p \in \mathbb{N}$.

5. Permanent properties from integer order derivatives

5.1. Kernels of linear operators with constant coefficients

To further substantiate the above used phrase "natural extension" we present in continuation of observation 3.2. the kernels of the operators $\mathcal{A}$.

Theorem 5.1. *Let $A(s)$ again denote the symbol of the linear operator $\mathcal{A}$. Let the principal branch zeros of $A(s)$ be $s_k = \sigma_k \pm i\nu_k$ with multiplicity m_k. Then the real solutions of*

$$\mathcal{A}x(t) = 0$$

are given by all $x(t)$:

$$x(t) = \sum_k e^{\sigma_k t}\Big(p_{m_k}(t)\,\sin(\nu_k t) + q_{m_k}(t)\,\cos(\nu_k t)\Big),$$

where p_{m_k}, q_{m_k} *are arbitrary real polynomials of degree* $\leq m_k - 1$.

For a proof see [21, theorem 14]. There it turns out that the absence of fractional parts in the kernel is based on the fact that fractional order derivatives of $\delta(\omega)$ are no point measures (see examples 4.1.4.). Thus, the shape of the kernels of fractional operators is just the same as in case of integer order operators.

The second essential feature which we now present concerns a central operation in *Fourier* analysis.

5.2. Convolutions

A problem similiar to the one we have discussed at the beginning of section 4.2. is the definition of convolutions in distributional spaces $\mathcal{K}'$. And similar to the definition of a product, a convolution of distributions does not exist in general. Trivially, if $\phi \in \mathcal{K}$, $f \in \mathcal{K}'$, then $\phi * f := \langle f\,,\, -\varphi * \phi\rangle$ (for all $\varphi \in \mathcal{K}$) exists and is moreover a $\mathbf{C}^\infty$–function. But the range of well–defined convolutions is much larger. The most popular result is (for proof see e.g. [26], 6.36, 6.37):

Theorem 5.2. *Let* $f \in \mathcal{E}'$, $g \in \mathcal{D}'$ *or* $g \in \mathcal{E}'$, $f \in \mathcal{D}'$.
Then the convolution $f * g = g * f \in \mathcal{D}'$ exists and is defined via:

$$(f * g) * \varphi := f * (g * \varphi) \qquad \text{for all } \ \varphi \in \mathcal{D}.$$

Now, the differentiation rule of convolutions can be extended with respect both to the differentiation order as well as to distributions by a very simple proof.

Theorem 5.3. *Let* $f * g \in \mathcal{K}'$, $q \in \mathbb{R}^+$. *Then it holds, as far as the resulting convolutions exist*

$$D^q\,(f * g) = (D^q f) * g = f * (D^q g)$$

P r o o f. $D^q\,(f * g) = \mathcal{F}^{-1}(i\omega)^q \mathcal{F}\,(f * g) = \sqrt{2\pi}\,\mathcal{F}^{-1}\Big\{(i\omega)^q \widehat{f}\widehat{g}\Big\} =$

$$= \sqrt{2\pi}\,\mathcal{F}^{-1}\Big\{\Big[(i\omega)^q \widehat{f}\Big]\,\widehat{g}\Big\} = \sqrt{2\pi}\,\mathcal{F}^{-1}\Big\{\widehat{(D^q f)}\,\widehat{g}\Big\} = (D^q f) * g \text{ ex.def.}$$

The second equation follows from the commutativity of $f * g$ or $\widehat{f}\widehat{g}$, resp., ■.

We end this paper by the announced link to the usual fractional calculus which is now an easy consequence of this theorem.

5.3. Riemann–Liouville and Caputo integrals

Obviously, we can try to calculate $D^q f = D^q\,(f * \delta) = f * (D^q \delta)$. But, whereas $D^q f$, $f \in \mathcal{K}'$ always exists, the last convolution may not exist, if $q \notin \mathbb{N}$. Let us see, what happens. Using formally the integral representation of convolutions one gets via example 4.1.,5.

$$f * (D^q \delta) = \int\limits_{-\infty}^{\infty} f(\tau)\,\frac{\theta(t)\;(t-\tau)^{-q-1}}{\Gamma(-q)}\,\mathrm{d}\tau = \frac{1}{\Gamma(-q)} \int\limits_{-\infty}^{t} \frac{f(\tau)}{(t-\tau)^{1+q}}\,\mathrm{d}\tau\ ,$$

which is very ill–natured, due to the strong singularity of the kernel. But as the derivative exists, there must be a way out.

1. The degree of the kernel should be > -1 to get a weak singularity, such that those integrals exist for a big class of functions f.
2. From example 4.1.,5. we see, that $f * (D^p\theta)$ has this property, if $0 < p < 1$.
3. Consequently, if $n-1 < q < n$, we use the semigroup property of our approach to rewrite

$$D^q f = D^{q+1}\left(f * \theta\right) \overset{(1)}{=} D^n\left(f * \left(D^{q-n+1}\theta\right)\right) \overset{(2)}{=} \left(D^n f\right) * \left(D^{q-n+1}\theta\right)$$

For all functions for which convolution (1) has an integral representation we have arrived at a *Riemann–Liouville* integral. If an integral representation of (2) is possible, then we have a *Caputo* integral. Thus we conclude

THEOREM 5.4. *Let $n-1 < q < n$, $n \in \mathbb{N}$.*
Then it holds, if the accordant integral exists

$$\begin{aligned} D^q f(t) \quad &\overset{(1)}{=} \ {}_{-\infty}D_t^q\, f(t) &&= \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int\limits_{-\infty}^{t} \frac{f(\tau)\, d\tau}{(t-\tau)^{q-n+1}} \\ &\overset{(2)}{=} \ {}_{-\infty}^{\ \ C}D_t^q\, f(t) &&= \frac{1}{\Gamma(n-q)} \int\limits_{-\infty}^{t} \frac{f^{(n)}(\tau)\, d\tau}{(t-\tau)^{q-n+1}} \end{aligned}$$

REMARKS 5.1.

1. The theorem is trivially true for $0 \le n-1 = q$.
2. Of course the scope of the two integrals is very different. Particularly, the *Caputo* integral (2) requires higher differentiabilty of f, whereas in case of the existence of the integral in (1) the right expression exists at least in terms of distributions.
3. The equivalence of both representations (if both exist) is in accordance with the above given difference formula (4), namely, $f^{(n)}(\tau)$ vanishes necessarily for $t \to -\infty$, such that the difference vanishes for $a \to -\infty$.

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